Don’t Underestimate Exponential Growth!

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In this talk, we will talk about one concept: **Exponential models**.

We will consider it in two different ways:

1. **Mathematics (obviously!)**
   - What is mathematical modelling?
   - What is an exponential growth model?
   - Where can exponential models be used?
   - What are the limitations of exponential models?

2. **Behavioural / Decision-making.**
   - What are common (mis)perceptions of exponential growth models?
   - What are the **costs** of these misperceptions?

Let’s dive in with some examples.
Rice and Chessboard problem

A common folk story:

A grain of rice is placed on a square of a chessboard. Then 2 grains are placed in the next square, 4 grains are placed in the one after, then 8, then 16 and so on...

How many grains of rice would there be on the chessboard? (64 squares)
Parents’ advice: Save Money Early!

If you put money in a bank, the bank pays interest to you. This is certain percentage of the amount in the bank. e.g. If a bank pays 6% interest per year and you put $100,000 into the account, after one year you would have

\[100000 + 0.06 \times 100000 = 106000\]

After two years, you would have

\[106000 + 0.06 \times 106000 = 112360\]

How much more money would you have when you retire (age 65) if you started saving at age 25 compared to age 35?
New Car

You really really want to buy the new car (price $400,000) but don't have money for it right away.

You decide to borrow money from a bank to make the purchase. Banks charge an amount of money yearly which is a certain percentage of the money that hasn't paid back yet (this is also called interest).

Suppose that the bank charges 10% interest (per year) on their loan. If you decide to pay $50,000 per year, what proportion of the loan would you have paid off after 5 years?
Time to vote!

In the next few slides, I’m going to get you to vote on the answers to the previous 3 questions. Don’t worry, it will be multiple choice and I will revisit the questions again.

Please use the QR code below

![QR Code](image)

or type http://bit.ly/E2MAT001T into your browser to access the Google Forms page.

I’ll give you a few minutes to get ready.
A grain of rice is placed on a square of a chessboard. Then 2 grains are placed in the next square, 4 grains are placed in the one after, then 8, then 16 and so on...

How many grains of rice would there be on the chessboard? (64 squares)

Vote on the answer you think is the closest.

(A) $10^4 = 10000$
(B) $10^8 = 100000000$
(C) $10^{12} = 1000000000000$
(D) $10^{16} = 10000000000000000$
(E) $10^{20} = 100000000000000000000$
A bank pays 6% interest per year and you put $100,000 into the account.

How much more money would you have when you retire (age 65) if you started saving at age 25 compared to age 35?

Vote on the answer you think is the closest.

(A) 40 \% more.
(B) 50 \% more.
(C) 60 \% more.
(D) 70 \% more.
(E) 80 \% more.
You decide to borrow $400,000 from a bank to buy a new car.

Suppose that the bank charges 10% interest (per year) on their loan and you decide to pay $50,000 per year, what proportion of the loan would you have paid off after 5 years?

Vote on the answer you think is the closest.

(A) 20 % of the loan.
(B) 30 % of the loan.
(C) 40 % of the loan.
(D) 50 % of the loan.
(E) 60 % of the loan.
Please wait while I compile the results!
Answer: Rice and Chessboard problem

A grain of rice is placed on a square of a chessboard. Then 2 grains are placed in the next square, 4 grains are placed in the one after, then 8, then 16 and so on...

How many grains of rice would there be on the chessboard? (64 squares)

Answer: (E) $10^{20} = 100000000000000000000$

From one square to the next, we multiply by 2 each time. We express this using powers of 2:

<table>
<thead>
<tr>
<th>Number of square</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grains of rice</td>
<td>1</td>
<td>$2^1$</td>
<td>$2^2$</td>
<td>...</td>
<td>$2^{63}$</td>
</tr>
</tbody>
</table>

Adding up the grains of rice, we get $1.8 \times 10^{19}$. 
Answer: Rice and Chessboard problem

It’s quite hard to understand exactly how big the number $1.8 \times 10^{19}$ is.

It is much much more than the number of grains of rice available in the entire world.

In fact it is more than the number of grains of sand on Earth!
Answer: Saving Money $$$

A bank pays 6% interest per year and you put $100,000 into the account.

How much more money would you have when you retire (age 65) if you started saving at age 25 compared to age 35?

Answer: (E): 80% more.

Note that a 6% increase corresponds to multiplying by 1.06 every year and saving from 25 to retirement is the same as saving for 40 years whilst saving from 35 to retirement is saving for 30 years.

\[
\begin{align*}
1 \text{ year: } & \quad 1.06 \times 100000 = \$106000 \\
2 \text{ years: } & \quad 1.06 \times 106000 = 1.06^2 \times 100000 = \$112360 \\
3 \text{ years: } & \quad 1.06^3 \times 100000 \approx \$119102 \\
\vdots & \quad \vdots \\
30 \text{ years: } & \quad 1.06^{30} \times 100000 \approx \$574000 \\
\vdots & \quad \vdots \\
40 \text{ years: } & \quad 1.06^{40} \times 100000 \approx \$1029000
\end{align*}
\]
Answer: Saving Money $$$

So saving for 33% more time (40 years instead of 30 years) gives you a 80% increase.

Conclusion: Listen to your parents!!

If you add money to the bank account on a yearly basis, the difference is more pronounced!

e.g., if you add $12000 to the account every year then

After 30 years: $1.58 million
After 40 years: $3 million

That’s a 90% increase!
Answer: Borrowing Money

You decide to borrow $400,000 from a bank to make the purchase. Suppose that the bank charges 10% interest (per year) on their loan and you decide to pay $50,000 per year, what proportion of the loan would you have paid off after 5 years?

Answer: (B) 30% of the loan

Remember that interest on the loan is calculated even as you are paying it off. Using the loan calculator from https://calculator.net

![Loan Calculator](https://calculator.net)

**Results:**
- Payment Every Year: $49,865.65
- Total of 17 Payments: $847,716.11
- Total Interest: $447,716.11

View Amortization Table
Conclusion:

忠告：借錢梗要還，咪俾錢中介

To borrow or not to borrow? Borrow only if you can repay!
All these situations are examples of **exponential models**. We can identify them because they involve taking successively higher powers.

An exponential model is of the form:

\[ y = a \cdot b^x \]

for numbers \( a > 0, \ b > 0 \).

Exponential models appear in many areas of life. We have just seen that they occur when there is repeated multiplication.
We have (hopefully) seen how the growth of exponential models are often underestimated. This is called **exponential growth bias**.

There are huge costs of exponential bias. One example we have seen is in personal finance.

- It can cause people to undervalue saving money.
- It can cause people to take risky loans because they underestimate how much money and how long it would take to repay a loan.

These can create huge problems at many levels.
What is bias?

In this talk, we consider bias as a deviation from rational judgement.

In other words, it is how a group of people’s belief might not match with the logical result.

Social psychologists study these biases because we often want to know how people will *actually* behave rather than what is the logical behaviour.

This has important consequences in many fields where assuming that people act logically and rationally can have consequences when the reality does not match the theory.

An example of such a field is **public health**.
Exponential bias in the Covid-19 Pandemic

We will see that that the spread of Covid-19 can be considered as an exponential model.

Exponential bias causes people to underestimate the spread of the virus. For example:

- Not following social distancing guidelines.
- Not wearing masks.
- etc...

The consequence is more people catching the virus and more people dying from it.
Recall the following rules for exponents:

\[ a^n \times a^m = a^{n+m} \quad a^n \div a^m = a^{n-m} \quad (a^n)^m = a^{nm} \quad a^0 = 1 \]

As you may know, you can consider negative or fractional exponents as well but we won’t be concerned with them.

We can plot the graph of \( y = a^x \) (for \( a > 1 \)): e.g. \( y = 2^x \):
What is mathematical modelling?

Mathematical modelling is the use of mathematics to describe real life situations.

Mathematical modelling plays a central role in

- Sciences
- Engineering
- Economics and Finance
- Social Science

Having a mathematical model allows us to make predictions.

However, it is not correct to say that the model fully represents the real life situations. To create a model, we must make assumptions in order to simplify things.
For future comparison purposes, we first look at linear models:

- \( y = mx + b \) e.g. \( y = 2x + 1 \).

- \( b \) is the initial value (\( y \)-intercept).
- As \( x \) increases by 1, \( y \) increases by \( m \).
Linear Models

![Graph showing linear models with equations: y = 4x, y = 3x, y = 2x, y = x, y = 0.5x.](image)

- $y = 4x$
- $y = 3x$
- $y = 2x$
- $y = x$
- $y = 0.5x$
We now look at exponential growth:

- \( y = a \cdot b^x \) where \( b > 0 \) and \( a > 0 \) e.g. \( y = 0.5 \cdot 2^x \).

- \( a \) is the initial value (since \( b^0 = 1 \)).
- As \( x \) increases by 1, \( y \) **multiplies** by \( b \).
Exponential Growth

When $b > 1$ in $y = a \cdot b^x$, we get exponential growth.
Exponential Decay

When $0 < b < 1$ in $y = a \cdot b^x$, we get exponential decay.
Logarithmic Scale

As can be seen from the graphs, it can be hard to plot exponential growth because it grows so quickly!

We can get around this by changing the scale on the y-axis:

\[
1 \mapsto 10, 2 \mapsto 10^2, 3 \mapsto 10^3, \ldots, n \mapsto 10^n
\]

This is called logarithmic scale. Exponential graphs become linear in this scale!
We can now say what is exponential growth bias in more clear terms:

What is exponential growth bias?

Exponential growth bias is the tendency to linearize exponential growth.
Examples

We have seen in the earlier examples where exponential growth appears.

The earlier example of saving money and borrowing money are examples of **compound interest**.

Interest is applied every year on the savings amount or loan amount. This corresponds to a repeated percentage change. In other words, it is repeated multiplication. This gives rise to an exponential model.

We now look at one more example: population growth.
Example: Population Growth

The growth of a population (humans, rabbits, bacteria etc.) was first described using an exponential model by Malthus (1766). He assumed that a population doubles after a certain fixed time period.

For example, a population initially has 100 individuals. Every 3 years, the population would double.

<table>
<thead>
<tr>
<th>Year</th>
<th>0</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population</td>
<td>100</td>
<td>200</td>
<td>400</td>
<td>800</td>
<td>1600</td>
<td>3200</td>
</tr>
</tbody>
</table>

Using the formula for an exponential model, $y = a \cdot b^x$, we get $a = 100$ (since $a$ is the initial value). Also,

$$a \times b^4 = 200 \Rightarrow b^4 = 200 \Rightarrow b = \sqrt[4]{2} = 1.19$$
Example: Population Growth

So $y = 100 \cdot 1.19^x$.

Malthus also observed that population growth cannot keep increasing exponentially. Resources (food, water, space etc.) do not grow as fast as the population. So eventually, the growth slows down.

The red graph is the exponential model. The blue graph is the modified version to take into account the reduced amount of resources as the population grows.
Modifying the model

The modified population growth model exhibits an exponential growth phase followed by a slowing down of the growth. This model is actually called **logistic growth**

How can we modify the model in order to obtain this?
Changing the multiplier

In the exponential model \( y = a \cdot b^x \), we saw that \( b \) acts as a multiplier. i.e. as \( x \) goes up by 1, \( y \) multiplies by \( b \).

We also saw:

- If \( b > 1 \), we have growth.
- If \( 0 < b < 1 \), we have decay.
- If \( b = 1 \), the value remains constant.

To modify the exponential model, we can allow \( b \) to change for different values of \( x \)
e.g. if \( b = 1 + \frac{1}{x} \), we would first multiply by 2, then by 1.5, then by 1.33 etc.
### $b = 1.5$

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>1</td>
<td>1.5</td>
<td>2.25</td>
<td>3.38</td>
<td>5.06</td>
<td>7.59</td>
<td>11.4</td>
<td>17.1</td>
<td>25.6</td>
</tr>
<tr>
<td>$b$</td>
<td>-</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
</tr>
</tbody>
</table>

### $b = 1 + \frac{5.5}{x^2}$

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>1</td>
<td>6.5</td>
<td>15.4</td>
<td>24.9</td>
<td>33.4</td>
<td>40.8</td>
<td>47.0</td>
<td>52.3</td>
<td>56.8</td>
</tr>
<tr>
<td>$b$</td>
<td>-</td>
<td>6.5</td>
<td>2.38</td>
<td>1.61</td>
<td>1.34</td>
<td>1.22</td>
<td>1.15</td>
<td>1.11</td>
<td>1.09</td>
</tr>
</tbody>
</table>

### $b = 0.5 + \frac{4}{x}$

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>15</td>
<td>20.6</td>
<td>24.8</td>
<td>26.8</td>
<td>26.8</td>
<td>25.1</td>
</tr>
<tr>
<td>$b$</td>
<td>-</td>
<td>4</td>
<td>2.25</td>
<td>1.67</td>
<td>1.38</td>
<td>1.2</td>
<td>1.08</td>
<td>1</td>
<td>0.94</td>
</tr>
</tbody>
</table>
Pandemic Models

We now consider the situation that we all find ourselves in: the spread of Covid-19.

Let $y$ be the number of people with Covid-19 and $x$ be days. As people infected with Covid-19 infect other people, we expect a percentage increase in people infected every day.

This suggests an exponential model:

$$y = a \times (1 + \beta - \gamma)^x$$

where $\beta$ is the infection rate and $\gamma$ is the recovery rate.

The multiplier is then $b = (1 + \beta - \gamma)$. If this is greater than one, we have exponential growth!
The R number

However, we also need to consider the number of susceptible people i.e. people that risk catching the disease.

The infection rate should be modified by the proportion of susceptible people in the population:

\[ b = (1 + \beta S(x) - \gamma) \]

Here \( S(x) \) is the proportion of the population that is still susceptible to the disease. Note that, \( S(x) \) changes as \( x \) changes (as people get infected, they are no longer susceptible).

The multiplier \( b \) is also called the R number. Scientists have predicted that if no action were taken to tackle the spread, the R number would be greater than 3.
What can be done?

To stop the spread and eventually eliminate the virus, we need to keep the $R$ number less than one so we DO NOT have exponential growth.

The form of the $R$ number, $(1 + \beta S(x) - \gamma)$ suggests what can be done.

- We can decrease $\beta$, the infection rate.
  e.g. by wearing masks, imposing limits on public gatherings and compulsory testing; implementing travel restrictions and quarantine.

- We can increase $\gamma$, the recovery rate.
  e.g. by conducting studies to find effective methods of treatment.

- We can decrease $S(x)$, the proportion of people susceptible.
  This is what vaccination achieves - an individual is no longer susceptible after vaccination.
Summary and conclusion

Today we have looked at

- What is exponential growth.
- What is exponential growth bias.
- What are the costs of exponential growth bias.
- Examples of exponential growth including the spread of COVID.

Conclusion

- Watch out for exponential growth bias!
- Follow government guidelines on limiting the spread of disease. Get the $R$ number less than one.
Thanks for listening!

I wrote an article on my personal website which goes into modelling the spread of diseases in more detail. Visit https://www.giftofmaths.com/epidemic.html

Feel free to email me at jtsai@giftofmaths.com if you have questions.

Please fill in the HKAGE evaluation: